WS 4.8: Exponential Growth & Decay; Newton's Law; Logistic Growth & Decay

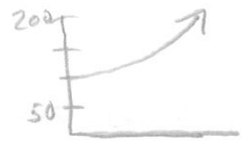
* Exponential Law or Law of uninhibited growth ($K>0$) or decay ($K<0$)
   - $A = A_0 e^{Kt}$, A_0 = original amt ($t=0$) AND $K \neq 0$ is a constant
     - ↳ If $K>0$, then A is increasing over time
     - ↳ If $K<0$, then A is decreasing over time

2.) a.) $N(0) = 100e^{0.045(0)} = \underline{100 \text{ grams}}$

b.) In $A = A_0 e^{Kt}$, K = rate as a decimal (convert to a percent)

$K = 0.045 \rightarrow \underline{4.5\%}$

c.) graph using a calc $\rightarrow$



d.) $N(5) = 100e^{0.045(5)} = \underline{125.2 \text{ grams}}$

e.) $\frac{140}{100} = \frac{\cancel{100}e^{0.045(t)}}{\cancel{100}} \rightarrow 1.4 = e^{0.045t}$

$\ln 1.4 = \ln e^{0.045t} \rightarrow \frac{\ln 1.4}{0.045} = \frac{0.045t}{0.045} \approx \boxed{7.5 \text{ days}}$

f.) Doubling Time general equation $\rightarrow 2P = Pe^{Kt}$

If $P = 100$, then $2P = 200$

$\frac{200}{100} = \frac{100e^{0.045t}}{100}$

$2 = e^{0.045t}$

$\ln 2 = \ln e^{0.045t}$

$\frac{\ln 2}{0.045} = \frac{\cancel{0.045}t}{\cancel{0.045}}$

$\boxed{t \approx 15.4 \text{ days}}$

- $N(t) = N_0 e^{Kt}$, $K>0$

where N_0 is the initial number of cells and K is a positive constant that represents the growth rate of the cells.

3.) A.) use $N(t) = N_0 e^{Kt} \rightarrow N(3) = 2N_0$

$$\frac{\cancel{N_0} e^{K(3)}}{\cancel{N_0}} = \frac{2\cancel{N_0}}{\cancel{N_0}} \rightarrow e^{3K} = 2 \rightarrow \ln e^{3K} = \ln 2$$

$$\frac{3K}{3} = \frac{\ln 2}{3}$$

$$K = \frac{\ln 2}{3}$$

$$\boxed{N(t) = N_0 \cdot e^{\left(\frac{\ln 2}{3}\right)t}}$$

B.) $$\frac{3\cancel{N_0}}{\cancel{N_0}} = \frac{N_0 e^{\left(\frac{\ln 2}{3}\right)t}}{N_0} \rightarrow 3 = e^{\left(\frac{\ln 2}{3}\right)t}$$

$$\ln 3 = \ln e^{\left(\frac{\ln 2}{3}\right)t}$$

$$\frac{\ln 3}{\frac{\ln 2}{3}} = \frac{\cancel{\left(\frac{\ln 2}{3}\right)} t}{\cancel{\frac{\ln 2}{3}}} \rightarrow \boxed{t \approx 4.755 \text{ hrs}}$$

or

$\boxed{4 \text{ hrs}, 45 \text{ mins}}$

C.) If a population doubles in 3 hours, then it will double a second time in 3 more hours, for a total of 6 hrs.

4.) Radioactive Decay $\rightarrow A(t) = A_0 e^{Kt}$, $K < 0$ (K = rate of decay)

① Find K $\rightarrow$ $$\frac{\frac{1}{2}\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0} e^{K(5600)}}{\cancel{A_0}} \rightarrow \frac{1}{2} = e^{5600K}$$

$$\ln \frac{1}{2} = \ln e^{5600K}$$

$$\frac{\ln \frac{1}{2}}{5600} = \frac{\cancel{5600}K}{\cancel{5600}}$$

$$\frac{\ln \frac{1}{2}}{5600} = K$$

② New formula (using 1.67%)

$$\frac{0.0167\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0} e^{\frac{\ln \frac{1}{2}}{5600}t}}{\cancel{A_0}}$$

$$0.0167 = e^{\frac{\ln \frac{1}{2}}{5600}t}$$

$$\ln 0.0167 = \ln e^{\frac{\ln \frac{1}{2}}{5600}t}$$

$$\frac{\ln 0.0167}{\frac{\ln \frac{1}{2}}{5600}} = \frac{\cancel{\frac{\ln \frac{1}{2}}{5600}} t}{\cancel{\frac{\ln \frac{1}{2}}{5600}}}$$

$$\boxed{t \approx 33{,}062 \text{ yrs}}$$

4.8 Notes Continued – Logistic Model; Newton's Law of Cooling

<u>Logistic Model</u>: Describes situations where the growth or decay of the dependent variable is limited. Ex- Population growth

In a logistic growth model, the population P after time t obeys the equation: $P(t) = \dfrac{c}{1+ae^{-bt}}$

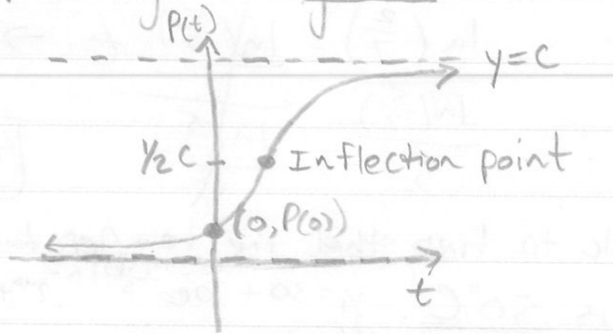
where a, b, + c are constants w/ $c > 0$.

- Growth model: If $b > 0$
- Decay model: If $b < 0$

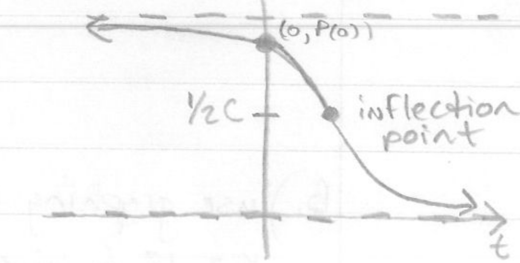
The number <u>c</u> is called the <u>carrying capacity</u> (for growth models) b/c $P(t)$ approaches c as $t \to \infty$.

That is: $\lim_{t\to\infty} P(t) = c$. The number $|b|$ is the <u>growth rate</u> for $b > 0$, and the decay rate for $b < 0$.

Typical logistical <u>growth</u> function



Typical logistical <u>decay</u> function



Newton's Law of Cooling

5.) The temperature u of a heated object at a given time t can be modeled by:

$$u(t) = T + (u_0 - T)e^{kt}, \quad k<0$$

where T is the constant temperature of the surrounding medium, u_0 is the initial temperature of the heated object, and k is a negative constant.

5.) A. $u(t) = 30 + (100-30)e^{kt} = 30 + 70e^{kt}$

to find k, sub in $u=80$ when $t=5$

$80 = 30 + 70e^{k(5)}$

$\frac{50}{70} = \frac{\cancel{70}e^{5k}}{\cancel{70}} \rightarrow \frac{5}{7} = e^{5k} \rightarrow \ln(\frac{5}{7}) = \ln e^{5k}$

$\frac{\ln(\frac{5}{7})}{5} = \frac{\cancel{5}k}{\cancel{5}} \rightarrow \boxed{k \approx -0.0673}$

$u(t) = 30 + 70e^{\frac{\ln(5/7)}{5}t} \rightarrow$ find t when $u = 50°C$

$50 = 30 + 70e^{\frac{\ln(5/7)}{5}t}$

$\frac{20}{70} = \frac{\cancel{70}e^{\frac{\ln(5/7)}{5}t}}{\cancel{70}} \rightarrow \frac{2}{7} = e^{\frac{\ln(5/7)}{5}t} \rightarrow \ln\frac{2}{7} = \ln e^{\frac{\ln(5/7)}{5}t}$

$\frac{\ln(\frac{2}{7})}{\frac{\ln(\frac{5}{7})}{5}} = \frac{\cancel{\frac{\ln(\frac{5}{7})}{5}}\,t}{\cancel{\frac{\ln(\frac{5}{7})}{5}}} \rightarrow \boxed{t \approx 18.6 \text{ mins}}$

or

$\boxed{t \approx 18 \text{ mins}, 37 \text{ sec}}$

B.) use graphing calc to find that the temperature at $x = 18.6$ mins is <u>50°C</u>. $y_1 = 30 + 70e^{\frac{\ln(5/7)}{5}x}$, 2nd trace → value → 18.6

C.) $y_1 = 35$, $y_2 = 30 + 70e^{\frac{\ln(5/7)}{5}x}$. use intersect function to find that it takes <u>x = 39.22 mins</u> for the temp to cool to 35°C.

D.) As x increases, $e^{\frac{\ln(5/7)}{5}x}$ approaches zero, so y (the temp of obj) approaches $\boxed{30°C}$.

$$P(t) = \frac{230}{1+56.5e^{-.37t}}$$

6.) A.) As $t \to \infty$, $e^{-.37t} \to 0$ AND $P(t) \to \frac{230}{1}$.

- The carrying capacity is $\boxed{230}$ Fruit Flies.
- The growth rate is $|b| = |0.37| = \boxed{37\%}$

B.) $P(0) = \frac{230}{1+56.5e^{-.37(0)}} = \frac{230}{1+56.5} = \boxed{4 \text{ fruit flies} \downarrow \text{initial population}}$

C.) $P(5) = \frac{230}{1+56.5e^{-.37(5)}} \approx \boxed{23 \text{ Fruit Flies}}$

D.) $180 = \frac{230}{1+56.5e^{-.37t}}$

* use graphing calc + find intersection $Y_1 = 180$, $Y_2 = \frac{230}{1+56.5e^{-.37t}}$

$$(1+56.5e^{-.37t}) \cdot 180 = \frac{230}{\cancel{1+56.5e^{-.37t}}} \cdot \cancel{(1+56.5e^{-.37t})}$$

$$\frac{(1+56.5e^{-.37t})\cancel{180}}{\cancel{180}} = \frac{230}{180}$$

$$1+56.5e^{-.37t} = 1.2778$$

$$56.5e^{-.37t} = 1.2778 - 1$$

$$\frac{\cancel{56.5}e^{-.37t}}{\cancel{56.5}} = \frac{0.2778}{56.5}$$

$$e^{-.37t} = 0.0049$$

$$\ln e^{-.37t} = \ln 0.0049$$

$$\frac{\cancel{-.37}t}{\cancel{-.37}} = \frac{\ln 0.0049}{-.37}$$

$$\boxed{t \approx 14.4 \text{ DAYS}}$$

E.) *use graphing calc + find the x-coordinate of the intersection point

$Y_1 = 115$ (half of 230)

$Y_2 = \frac{230}{1+56.5e^{-.37t}}$

$t \approx 10.9$ DAYS

or

$t \approx 10$ DAYS, 22 hrs

7.) $P(t) = \dfrac{100.3952}{1 + 0.0316e^{0.0581t}}$

A.) Decay rate is $|b| = |-0.0581| = \boxed{5.81\%}$

B.) $P(10) = \dfrac{100.3952}{1 + 0.0316e^{0.0581(10)}} \approx \boxed{95.0}$

so 95% of wood products remain after 10 yrs

C.) * use graphing calc and find x-coordinate of the intersection point

$y_1 = 50$

$y_2 = \dfrac{100.3952}{1 + 0.0316e^{0.0581t}}$

$\boxed{t \approx 59.6 \text{ yrs}}$

- It will take approximately 59.6 yrs for the percentage of wood products to reach 50%.